Convergent perturbation theory for a q–deformed anharmonic oscillator

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Abstract. A q–deformed anharmonic oscillator is defined within the framework of q–deformed quantum mechanics. It is shown that the Rayleigh–Schrödinger perturbation series for the bounded spectrum converges to exact eigenstates and eigenvalues, for q close to 1. The radius of convergence becomes zero in the undeformed limit.

1 Introduction

The anharmonic oscillator $H = \omega a^\dagger a + \gamma X^4$ is a basic quantum mechanical problem with one particularly interesting feature: its perturbation series diverges, but nevertheless there exist eigenstates and energies which are smooth as the (positive) coupling constant γ goes to zero [1–3]. A similar phenomenon is expected to occur in many interacting quantum field theories. The anharmonic oscillator can in fact be considered as a $(0 + 1)$–dimensional φ^4 "field" theory with one degree of freedom.

In this paper, we study the analog of this model in the framework of q–deformed quantum mechanics, based on the q–deformed Heisenberg algebra introduced in [4]. In particular, one would like to know how the perturbation theory of the q–deformed anharmonic oscillator behaves compared to the undeformed case. This is of interest in view of a possible q–deformation of field theory, which is expected to be less singular than field theory based on ordinary manifolds, since q–deformation generically puts physics on a q–lattice [4,5]. With this motivation, we study the perturbation theory of the anharmonic oscillator in terms of the q–deformed harmonic oscillator, which was introduced in [6,7] and realized in the framework of q–deformed quantum mechanics in [8].

There is considerable freedom in defining a q–deformed anharmonic oscillator for $q \neq 1$. Taking advantage of this freedom, we show that for a suitable definition of the anharmonic oscillator, the perturbation series converges to exact eigenvalues and eigenstates for $1 < q < 1.06$ with a certain radius of convergence in γ. In the limit $q \to 1$, the model reduces to the usual anharmonic oscillator, and the radius of convergence goes to zero. The upper limit on q is not significant.

This paper is organized as follows: In Sect. 2 we review the q-deformed harmonic oscillator and its spectrum, and calculate the relevant matrix elements. In Sect. 3, the perturbation series for eigenvalues and eigenstates is discussed. Some estimates for the matrix elements are given in the Appendix.

2 The q-deformed harmonic oscillator

In this section, we give a brief review of the q–deformed harmonic oscillator, and its realization in terms of a q–deformed Heisenberg algebra. For a more detailed discussion, see [8] and [4].

The q-deformed Heisenberg algebra is the star–algebra generated by X, P, U with the relations [4]

$$q^{\frac{1}{2}} X P - q^{-\frac{1}{2}} P X = iU \tag{1}$$
$$UX = q^{-1} XU, \quad UP = qPU.$$

We assume $q > 1$ to be real. The star structure is such that X and P are hermitian, and U is unitary:

$$X = X^\dagger, \quad P = P^\dagger, \quad U^\dagger = U^{-1}. \tag{2}$$

This algebra has the following (momentum–space) representation [4]:

$$P|n,\sigma\rangle = \sigma q^n |n,\sigma\rangle$$
$$U|n,\sigma\rangle = |n-1,\sigma\rangle$$
$$U^{-1}|n,\sigma\rangle = |n+1,\sigma\rangle$$
$$X|n,\sigma\rangle = i\sigma \frac{q^{-n}}{q - q^{-1}}(q^{\frac{1}{2}}|n-1,\sigma\rangle - q^{-\frac{1}{2}}|n+1,\sigma\rangle)$$
$$\langle n,\sigma|m,\sigma'\rangle = \delta_{n,m}\delta_{\sigma,\sigma'} \tag{3}$$

with $n, m \in I\!N$ and $\sigma, \sigma' = \pm 1$. The completion of these states defines a Hilbert space $\mathcal{H}$.

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The two values of σ describe positive respectively negative momenta. (3) is a star–representation, i.e. the star is implemented as the adjoint of an operator, and both X and P have selfadjoint extensions. That is the reason for introducing σ, see [9].

This is a starting point for studying q–deformed quantum mechanics [10–12,4]. In particular, one can define q–deformed creation and anihilation operators as follows:

$$a = \alpha U^{-2M} + \beta U^{-M} P \tag{4}$$
$$a^\dagger = \bar{\alpha} U^{2M} + \bar{\beta} P U^M$$

with $M \in I\!N$, and $\alpha, \beta \in \mathbb{C}$. They satisfy the Biedenharn–Macfarlane algebra [6,7]:

$$a a^\dagger - q^{-2M} a^\dagger a = (1 - q^{-2M}) \alpha \bar{\alpha} = 1 \tag{5}$$

where we fix $\alpha = \frac{i}{\sqrt{1-q^{-2M}}}$. The occupation number operator is defined as

$$\hat{n} = a^\dagger a = \alpha \bar{\alpha} + \beta \bar{\beta} P^2 + \alpha \bar{\beta} (U^M + q^M U^{-M}) P. \tag{6}$$

Now one can write down the following Hamiltonian, which constitutes the q–deformed harmonic oscillator:

$$H_0 = \omega a^\dagger a \tag{7}$$

The spectrum of H_0 acting on $\mathcal{H}$ consists of a bounded spectrum with eigenvalues $E_n^{(0)} = \omega [n]_M = \omega \frac{1-q^{-2nM}}{1-q^{-2M}}$ which is $2M$–fold degenerate, and an unbounded spectrum with eigenvalues $\omega(q^{2mM} E_0^{(0)} + \frac{1-q^{2mM}}{1-q^{-2M}})$. The $2M$ ground states of the bounded spectrum are

$$|0\rangle_{\sigma,\mu}^{(M)} = \sum_{n=-\infty}^{\infty} c_0 \left(-\sigma \frac{\alpha}{\beta}\right)^n$$
$$\times q^{-\frac{1}{2}(Mn^2 + Mn + 2\mu n)} |Mn + \mu, \sigma\rangle,$$
$$0 \le \mu < M. \tag{8}$$

The existence of an unbounded spectrum beyond $E_\infty = \frac{\omega}{1-q^{-2M}}$ is clear in view of (5), since P is an unbounded operator on $\mathcal{H}$. For simplicity, we will only consider $M = 1$ from now on, and omit the labels μ and M.

So far, β was arbitrary. Requiring that the $a, a^\dagger$ are smooth for $q \to 1$ and become the usual (undeformed) creation and anihilation operators in the limit, one finds [8] that

$$\alpha = \frac{i}{\sqrt{1-q^{-2}}}, \quad \beta = \frac{i}{\sqrt{2m\omega}} \tag{9}$$

where m is the mass. For this choice, H_0 can be interpreted as a q–deformation of the usual harmonic oscillator, and this will be understood in the following. The normalized states of the bounded spectrum are

$$|n\rangle_\sigma = \frac{1}{\sqrt{[n]}} (a^\dagger)^n |0\rangle_\sigma, \tag{10}$$

where $[n] = \frac{1-q^{-2n}}{1-q^{-2}}$. We define $\mathcal{H}^{b,\pm} \subset \mathcal{H}$ to be the closure of the space spanned by the $|n\rangle_{\pm 1}$. As $q \to 1$, $\mathcal{H}^{b,+}$ becomes the Hilbert space of the usual harmonic oscillator, while the unbounded spectrum disappears at infinity, and the support of the states with $\sigma = -1$ goes to $-\infty$ in the momentum representation. We will thus concentrate on $\mathcal{H}^{b,+}$.

The eigenstates of H_0 can also be written in terms of the q–deformed Hermite polynomials, which satisfy (see [13]):

$$\xi H_n^{(q)}(\xi) = \frac{\sqrt{q} q^{2n}}{2} (H_{n+1}^{(q)}(\xi) + 2q^{-2}[n] H_{n-1}^{(q)}(\xi)) \tag{11}$$

Defining $\xi = \sqrt{m\omega} X$, one has

$$|n\rangle_\sigma = \frac{1}{\sqrt{2^n [n]!}} H_n^{(q)}(\xi) |0\rangle_\sigma.$$

Using these Hermite polynomials, it is straightforward to calculate the action of X on an eigenstate $|n\rangle_\sigma$, and it follows in particular that $X \cdot \mathcal{H}^{b,+} \subset \mathcal{H}^{b,+}$. This will be important for the perturbation theory below.

Now we turn to the anharmonic oscillator. The undeformed anharmonic oscillator is defined by $H = \omega a^\dagger a + \gamma X^4$ for $\gamma > 0$, thus one might naively take the same expression for $q > 1$, and study its perturbation theory. The relevant matrix elements can be calculated e.g. using (11), and we find the following results [14]:

$$\langle n | X^4 | n \rangle = \left(\frac{1}{2m\omega}\right)^2 q^{8n+6} ([n+1]([n+2]$$
$$+ q^{-4}[n+1] + q^{-8}[n]) q^{-8}[n]([n+1]$$
$$+ q^{-4}[n] + q^{-8}[n-1]))$$

$$\langle n+4 | X^4 | n \rangle = \left(\frac{1}{2m\omega}\right)^2 q^{8n+14}$$
$$\times \sqrt{[n+1][n+2][n+3][n+4]}$$

$$\langle n+2 | X^4 | n \rangle = \left(\frac{1}{2m\omega}\right)^2 q^{8n+12} \sqrt{[n+1][n+2]}([n+3]$$
$$+ q^{-4}[n+2] + q^{-8}[n+1] + q^{-12}[n]) \tag{12}$$

They are independent of σ which is suppressed. All other nonvanishing matrix elements can be obtained from those by hermiticity.

Looking at the powers of q in the matrix elements, one quickly finds that the perturbation series diverges even faster than in the undeformed case.

However, it is important to realize that there is no reason for considering the *same expression* for H as in the undeformed case; the only requirement one has to impose is that H should reduce to the usual anharmonic oscillator as $q \to 1$. Therefore we might just as well consider the Hamiltonian

$$H = H_0 + \gamma H' \tag{13}$$

with

$$H' = \frac{1}{2}(X^4 \hat{Q}^5 + \hat{Q}^5 X^4), \quad \text{where}$$
$$\hat{Q} = (1 - a^\dagger a(1 - q^{-2})). \tag{14}$$

$\hat{Q}$ satisfies

$$\hat{Q}|n\rangle = q^{-2n}|n\rangle. \tag{15}$$

The matrix elements $\langle n|H'|m\rangle$ can be easily obtained from (12), see Fig. 2. As is shown in the Appendix, they have the following upper bound:

$$\langle n|H'|m\rangle < C(q) := [3]_4[2]_8 q^{-2n_{max}+10}[n_{max}]^2$$
$$= \frac{4q^{10}[3]_4[2]_8}{81(1-q^{-2})^2} \tag{16}$$

for $1 < q < 1.06$, where $n_{max} = \frac{\ln 3}{2\ln q}$. In view of the results of the next section, we define (13) to be the q–deformed anharmonic oscillator.

3 Perturbation expansion

We will use the standard Rayleigh-Schrödinger perturbation formulas for the eigenstates and eigenvalues of

$$H = H_0 + H_1 = H_0 + \gamma H' \tag{17}$$

in terms of the unperturbed ones, $H_0|n\rangle = E_n^{(0)}|n\rangle$:

$$\Delta E_n = \sum_{k=0}^{\infty} E_n^{(k)}(\Delta E_n, \gamma) \tag{18}$$

$$:= \sum_{k=0}^{\infty} \langle n|H_1 \left(\frac{1}{E_n^{(0)} - H_0} Q_n (H_1 - \Delta E_n) \right)^k |n\rangle$$

where $Q_n = (1 - |n\rangle\langle n|)$, and

$$|E_n\rangle = |n\rangle + \sum_{k=1}^{\infty} \sum_{\substack{n_1,\ldots n_k \\ n_r \neq n}}$$

$$\times \frac{|n_1\rangle \left(\prod_{j=2}^{k} \langle n_{j-1}|H_1 - \Delta E_n|n_j\rangle \right) \langle n_k|H_1|n\rangle}{\prod_{j=1}^{k} (E_n^{(0)} - E_{n_j}^{(0)})}.$$

$$\tag{19}$$

Strictly speaking, we are of course dealing with a degenerate problem (since $\sigma = \pm 1$); however as already explained, X and $\hat{Q}$ leave $\mathcal{H}^{b,+}$ invariant, thus the two values of σ do not interfere, and we can restrict ourselves to the $\sigma = +1$ sector. This will be understood in the following. We will show that these series in fact converge to exact eigenvalues and eigenstates of the q-deformed anharmonic oscillator, for a certain range of γ which depends on q.

3.1 Energy levels

If γ and ΔE_n are not real, then H_1 is understood to act on the right in the above formulas, so that the matrix elements can be continued analytically in γ and ΔE_n. We

show first that the sum in (19) is absolutely convergent for $|\Delta E_n| < \omega/5$ and $|\gamma| < \gamma(q)$, where $\gamma(q) > 0$ provided $q > 1$, see (22). Thus the rhs of (19) is an analytic function of ΔE_n and γ in that domain, which can be solved for ΔE_n by the implicit function theorem, defining an analytic function $\Delta E_n(\gamma)$.

To see that the sum in (19) is (absolutely) convergent for a certain range of ΔE_n and γ, we first write $E_n^{(m)}$ more explicitly:

$$E_n^{(1)} = \langle n|H_1|n\rangle$$
$$E_n^{(2)} = \sum_{\substack{n_1 \\ n_1 \neq n}} \frac{\langle n|H_1|n_1\rangle\langle n_1|H_1|n\rangle}{(E_n^{(0)} - E_{n_1}^{(0)})}$$

$$E_n^{(k)}(\Delta E_n, \gamma) = \sum_{\substack{n_1,n_2,\ldots,n_{k-1} \\ n_r \neq n}}$$

$$\times \frac{\langle n|H_1|n_1\rangle \left(\prod_{j=2}^{k-1} \langle n_{j-1}|H_1 - \Delta E_n|n_j\rangle \right) \langle n_{k-1}|H_1|n\rangle}{(E_n^{(0)} - E_{n_{k-1}}^{(0)})(E_n^{(0)} - E_{n_{k-2}}^{(0)})\ldots(E_n^{(0)} - E_{n_1}^{(0)})}$$
$$\text{for } k \geq 3 \tag{20}$$

As is shown in Appendix A, the following estimate is valid for $q \in]1; 1.06[$:

$$|E_n^{(k)}(\Delta E_n, \gamma)| < \bar{E}_n^{(k)}(\Delta E_n, \gamma, q)$$
$$:= \frac{(|\gamma|C(q))^2(|\gamma|C(q) + |\Delta E_n|)^{k-2}5^{k-1}}{([2]\omega q^{-2n})^{k-1}}$$
$$\text{for } k \geq 2, \tag{21}$$

The factor 5 comes from the fact that for any given n_j, there are only 5 possible n_{j+1} such that the matrix elements in the perturbation expansion do not vanish (see (12)).

The series (19) is absolutely convergent if the following condition holds:

$$\left| \frac{\bar{E}_n^{(k+1)}}{\bar{E}_n^{(k)}} \right| < \theta \quad \text{for some} \quad \theta < 1$$

Now

$$\left| \frac{\bar{E}_n^{(k+1)}}{\bar{E}_n^{(k)}} \right| = 5\frac{|\gamma|C(q) + |\Delta E_n|}{[2]\omega q^{-2n}} < \theta,$$

and we find that the condition holds e.g. for $|\Delta E_n| < \omega/5$ and

$$|\gamma| \leq \gamma(q) := \frac{\omega([2]q^{-2n} - 1)}{5C(q)}. \tag{22}$$

Therefore we have shown that in this domain, the rhs of (18) defines an analytic function in ΔE_n and γ. Notice that $\gamma(q) \to 0$ as $q \to 1$.

Now consider the equation

$$G(\Delta E_n, \gamma) := \sum_{k=0}^{\infty} E_n^{(k)}(\Delta E_n, \gamma) - \Delta E_n = 0.$$
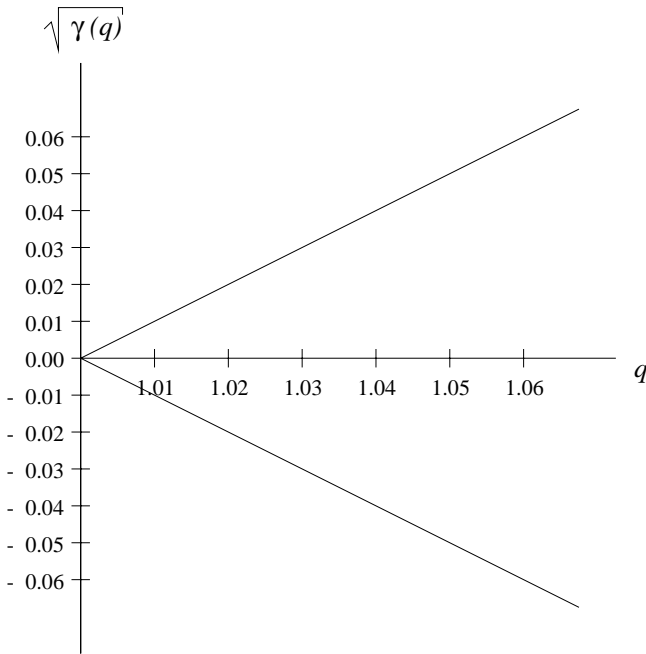
Fig. 1. Domain of convergence, for $\omega = 1$

In the above domain, this is a uniformly convergent series of analytic functions (for fixed q in the interval $]1, 1.06[$, say). But then using (20), one sees that $\frac{\partial}{\partial \Delta E_n} \sum_{k=0}^{\infty} E_n^{(k)}(\Delta E_n, \gamma)\Big|_{\substack{\Delta E_n = 0 \\ \gamma = 0}} = 0$, i.e.

$$\frac{\partial}{\partial \Delta E_n} G(\Delta E_n, \gamma) \neq 0 \qquad (23)$$

for γ and ΔE_n in a neighborhood of 0, by analyticity. Now the implicit function theorem states that there is a function $\Delta E_n(\gamma)$ which solves (23) and satisfies $\Delta E_n(0) = 0$. Moreover, $\Delta E_n(\gamma)$ is analytic in a neighborhood of 0, and $|\Delta E_n| < \omega/5$ holds automatically if γ is small enough.

The domain of convergence $\gamma(q)$ is shown in Fig. 1 for $q \in]1; 1.06[$ and $\omega = 1$. In particular, $\gamma(q)$ goes to zero for $q \to 1$, in accordance with the well–known fact that the perturbation series for the undeformed anharmonic oscillator is divergent [2].

3.2 Eigenstates

In this section, we show that (19) converges in $\mathcal{H}^{b,+} \subset \mathcal{H}$ for $|\gamma| < \gamma(q)$ and $1 < q < 1.06$, where $\Delta E_n = \Delta E_n(\gamma)$ is now the perturbed energy found in the previous section. To do this, we have to show that

$$\sum_{m=0}^{\infty} |\langle m | E_n \rangle|^2 < \infty, \qquad (24)$$

or more explicitly

$$\langle E_n | E_n \rangle = \sum_{m=0}^{\infty} |\langle m | E_n \rangle|^2$$

$$= \sum_{m=0}^{\infty} \left| \delta_{m,n} + \sum_{k=1}^{\infty} \sum_{\substack{n_1, \ldots, n_k \\ n_r \neq n}} \right.$$

$$\left. \delta_{m,n_1} \frac{\left(\prod_{j=2}^{k} \langle n_{j-1} | H_1 - \Delta E_n | n_j \rangle \right) \langle n_k | H_1 | n \rangle}{\prod_{j=1}^{k} (E_n^{(0)} - E_{n_j}^{(0)})} \right|^2$$

From the form of the matrix elements (12), we see that the second term is nonzero only for $k \geq \frac{|m-n|}{4}$, therefore

$$\langle E_n | E_n \rangle \leq 1 + \sum_{m=0}^{\infty} \sum_{k \geq \frac{|m-n|}{4}}$$

$$\times \left(\frac{(|\gamma| C(q) + |\Delta E_n|)^{k-1} |\gamma| C(q) 5^k}{([2] q^{-2n} \omega)^k} \right)^2$$

$$\leq 1 + \sum_{m=0}^{\infty} \left(5 \frac{|\gamma| C(q) + |\Delta E_n|}{[2] q^{-2n} \omega} \right)^{\frac{|m-n|}{2}}$$

$$\times \left(\sum_{k=0}^{\infty} \left(5 \frac{|\gamma| C(q) + |\Delta E_n|}{[2] q^{-2n} \omega} \right)^{2k} \right)$$

for $1 < q < 1.06$. Clearly this converges for γ in the analyticity domain defined above (such that $|\Delta E_n| < \omega/5$ as before), therefore the series (19) converges in $\mathcal{H}^{b,+}$. Finally, both H_0 and H' leave $\mathcal{H}^{b,+} \subset \mathcal{H}$ invariant and are bounded operators on $\mathcal{H}^{b,+}$ (H' is bounded because of (27) and the fact that H' acting on $|n\rangle$ has no more that 5 nonvanishing components in terms of that basis). Now it follows that $|E_n\rangle$ and $E_n^{(0)} + \Delta E_n$ are indeed eigenstates and eigenvalues of the full anharmonic oscillator.

As already mentioned, it is known [2] that the undeformed anharmonic oscillator does have nonperturbative eigenstates and energies for $\gamma > 0$, which are nevertheless smooth as γ goes to zero from above. Now the formulas (19) ff. can be analytically continued in q as well, and one would expect that the above domain of analyticity for ΔE_n and γ can be extended to include $q = 1$ and positive real axis of γ. However, at present we are not able to show this.

Appendix: Matrix elements

Because $[n]$ is an increasing function in n, we have the following estimates:

$$\frac{1}{2} \gamma (1 + q^{-40}) q^{14-2n} [n]^2$$
$$< \langle n + 4 | H' | n \rangle$$
$$< \frac{1}{2} \gamma (1 + q^{-40}) q^{14-2n} [n+4]^2$$
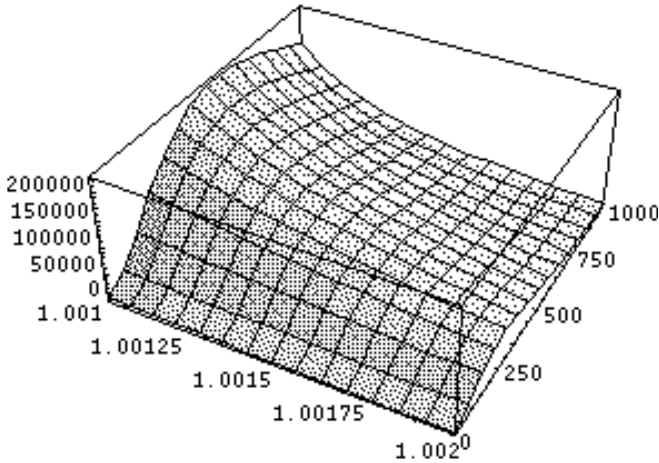
Fig. 2. The matrix elements $\langle n|H'|n\rangle$ for $q \in [1.001, 1.002]$ depending on n

$$\frac{1}{2}\gamma(1 + q^{-20})q^{12-2n}[4]_4[n]^2$$
$$< \langle n+2|H'|n\rangle$$
$$< \frac{1}{2}\gamma(1 + q^{-20})q^{12-2n}[4]_4[n+3]^2$$
$$\gamma q^{-2n+6}[3]_4[2]_8[n]^2$$
$$< \langle n|H'|n\rangle$$
$$< \gamma q^{-2n+6}[3]_4[2]_8[n+2]^2 \qquad (25)$$

with

$$[n]_i := \frac{1 - q^{-ni}}{1 - q^{-i}}, \quad [n] = \frac{1 - q^{-2n}}{1 - q^{-2}},$$

See Fig. 2 for a plot of $\langle n|H'|n\rangle$.

To simplify this, consider the function $q^{-2n}[n]^2$ for $n \in \mathrm{I\!R}$, which takes its maximum value $\frac{4}{81(1-q^{-2})^2}$ at $n = n_{max}$,

$$n_{max} := \frac{\ln 3}{2\ln q}. \qquad (26)$$

The matrix elements have a maximum for n close to n_{max}. More precisely, we can show the following estimate:

$$|\langle n+i|H'|n\rangle| < C(q) := q^{-2n_{max}+10}[3]_4[2]_8[n_{max}]^2$$
$$= \frac{4q^{10}[3]_4[2]_8}{81(1 - q^{-2})^2} \qquad (27)$$

for all $n, m \in \mathrm{I\!N}$. Indeed,

$$\langle n+4|H'|n\rangle < \frac{1}{2}(1 + q^{-40})q^{14-2n}[n+4]^2$$
$$= \frac{1}{2}(1 + q^{-40})q^{22}q^{-2(n+4)}[n+4]^2$$
$$\leq \frac{1}{2}(1 + q^{-40})q^{22}q^{-2n_{max}}[n_{max}]^2,$$

furthermore

$$\langle n+2|H'|n\rangle < \frac{1}{2}(1 + q^{-20})q^{18}[4]_4q^{-2(n+3)}[n+3]^2$$

$$\leq \frac{1}{2}(1 + q^{-20})q^{18}[4]_4q^{-2n_{max}}[n_{max}]^2, \qquad (28)$$

and

$$\langle n|H'|n\rangle \leq q^{10}[3]_4[2]_8q^{-2n_{max}}[n_{max}]^2 = C(q) \qquad (29)$$

Now for $1 \leq q < 1.06$, one has

$$1 < \frac{2q^{-16}[3]_4[2]_8}{1 + q^{-40}} \qquad (30)$$

(for $i = 4$) and

$$1 < \frac{2[3]_4[2]_8}{q^{12}[4]_4(1 + q^{-20})} \qquad (31)$$

(for $i = 1$). Combining these estimates, we obtain (27). Furthermore $|E_n^{(0)} - E_{n\pm i}^{(0)}| \geq [i]q^{-2n}\omega$, therefore $|E_{n_j}^{(0)} - E_n^{(0)}| \geq [2]q^{-2n}\omega$ in the denominators of the perturbation expansion, since $i \geq 2$. Now (21) follows, because for any given n_j in the perturbation series, there are at most 5 possible n_{j+1} such that $\langle n_j|H'|n_{j+1}\rangle$ is nonzero; this means that the number of terms at order k is at most 5^{k-1}.

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